REFERENCES

1. Krichevskii, I. Ia., On the accumulation of perturbations in linear systems. Dopovidi Akad. Nauk Ukrain. RSR, №8, 1966.
2. Krichevskii, I. Ia., On accumulation in nonlinear systems. In: Theory of Functions, Functional Analysis and Their Applications, №3, Izd. Khar'kovsk. Univ., 1967.
3. Krichevskii, I. Ia., On the effective solution of the problem of maximizing a linear functional. In: Theory of Functions, Functional Analysis and Their Applications, №4, Izd. Khar'kovsk. Univ., 1967.
4. Bulgakov, B. V., On the accumulation of perturbations in oscillatory systems with constant parameters. Dokl. Akad. Nauk SSSR Vol. 51, №5, 1946.
5. Bulgakov, B. V., Oscillations. Moscow-Leningrad, Gostekhizdat, 1949.
6. Lur'e, A. I., Operational Calculus and its Applications to Problems in Mechanics. Moscow, Gostekhizdat, 1957.
7. Ulanov, G. M., On the maximum error of a controlled variable in the transient response. Avtomat. i Telemechan. Vol. 9, №3, 1948.
8. Ulanov, G. M., Dynamic Accuracy and Perturbation Compensation in Automatic Control Systems. Moscow, "Mashinostroenie", 1971.
9. Gnoenskii, L. S., On the accumulation of disturbances in linear systems. PMM Vol. 25, №2, 1961.

Translated by N. H. C.

UDC 538.4

USE OF THE VARIATIONAL EQUATION IN THE STUDY OF POLARIZABLE AND MAGNETIZABLE CONTINUOUS MEDIA

PMM Vol. 37, №5, 1973, pp. 803-812

A. G. TSYPKIN
(Moscow)

(Received May 28, 1973)

A model of a continuum is constructed, using the variational equation suggested in [1, 2] which makes it possible to obtain models of continua using a minimum number of unified physical hypotheses. In the present paper the variational equation is used to obtain a system of equations defining the macroscopic motion of a continuum with polarization and magnetization effects taken into account, within the framework of the special relativity theory. Use of the four-dimensional space-time and special relativity theory is required in order to match theories of electromagnetism and mechanics. We investigate some of the consequences of two possible decompositions of the total energy-momentum tensor of the electromagnetic field and the continuum into the continuum energy-momentum tensor and the electromagnetic field energy-momentum tensor according to Minkowski and to Abraham, respectively. When moment stresses and external mass moments are absent in the medium, we assume the symmetry of the total energy-momentum tensor of electromagnetic field and medium (this is equivalent to the absence or constancy of the combined electromagnetic

field and medium, intrinsic internal moments of momenta along the world lines) and on this assumption consider certain models of continua.

1. Basic equations. As the basis for obtaining a consistent system of equations comprising the Maxwell equations and the equations of continuum mechanics, we use the variational equation in the form [1, 2]

$$\delta \int_{V_4} \Lambda d\tau_4 + \delta W^* + \delta W = 0 \tag{1.1}$$

where V_4 denotes an arbitrary four-dimensional volume of the pseudo-Euclidean Minkowski space-time, whose volume element

$$d\tau_4 = \sqrt{-g}\, dx^1 dx^2 dx^3 dx^4, \ g = \det \| g_{ij} \|$$

Here Λ is a Lagrange function which is a four-dimensional scalar, δW^* is a functional specified below, δW is a functional which is determined by specifying Λ and δW^*, and g_{ij} are the covariant components of the metric tensor of the observer's inertial coordinate system in a four-dimensional pseudo-Euclidean space. Further, as the four-dimensional Cartesian coordinate system we choose a coordinate system with the metric $ds^2 = -(dx^1)^2 - (dx^2)^2 - (dx^3)^2 + c^2 dt^2$ where c is the speed of light in vacuo and ds is an element of the arc length in the four-dimensional pseudo-Euclidean space. Unless otherwise stated, here and in the following, the lower case Italics indices assume the values 1, 2, 3, 4, the lower case Greek letter indices assume the values 1, 2, 3 and summation is performed over identical upper and lower indices.

Let us specify the total Lagrangian Λ of the electromagnetic field and continuous medium, in the following form:

$$\Lambda = -\frac{1}{16\pi} F^{ik} F_{ik} + \frac{1}{2} F_{ik} P^{ik} - \rho U(\rho, x^i{}_j, \pi^{ij}, g_{ij}, S, K_B) \tag{1.2}$$

Here $x_j{}^i = \partial x^i (\xi^k) / \partial \xi^j$: $x^i = x^i (\xi^k)$ is the law of motion of the medium, while ξ^1, ξ^2, ξ^3, $\xi^4 = \tau$ are the coordinates of points of the medium in a coordinate system frozen into the continuous medium and moving with it. [In what follows this system is called "intrinsic"]. In this coordinate system the metric is determined by

$$ds^2 = g_{\hat{i}\hat{j}} d\xi^i d\xi^j, \ g_{\hat{i}\hat{j}} = g_{\hat{i}\hat{j}} (\xi^k)$$

The superscript ^ denotes tensor components relative to the coordinate system and F_{ik} denote covariant components of the four-dimensional tensor of an electromagnetic field in the medium. These components are connected with the components of the four-dimensional vector potential A_k by the formulas

$$F_{ik} = \nabla_i A_k - \nabla_k A_i$$

where ∇_k is the covariant differentiation operator in the observer's coordinate system, P^{ij} denote the components of the polarization-magnetization tensor computed for a unit volume of the medium, $\pi^{ij} = \rho^{-1} P^{ij}$ by definition, where π^{ij} are the components of the polarization-magnetization tensor computed for a unit mass of the medium and ρ is the medium mass density defined by the formula

$$\rho = f(\xi^\mu) [\det \| g_{\hat{\alpha}\hat{\beta}} - u_{\hat{\alpha}} u_{\hat{\beta}} \|]^{-1/2} \tag{1.3}$$

Here $u_{\hat{i}}$ are the covariant components of the four-velocity vector of the medium relative to the observer's coordinate system computed in the comoving coordinate system (the four-velocity vector is determined in the inertial coordinate system by its components $u^i = (dx^i/d\tau)_{\xi^k=\text{const}}$) . S is the entropy per unit mass of the medium and U is the internal energy of a unit mass of the medium.

Let us introduce the quantity ρ_e denoting the medium free electrical charge density

$$\rho_e = \varphi(\xi^\mu)\,[\det \| g_{\hat{\alpha}\hat{\beta}} - u_{\hat{\alpha}} u_{\hat{\beta}} \|]^{-1/2} \qquad (1.4)$$

In accordance with the definition of the free electrical charge density and of the four-velocity vector of the medium, the quantity $\rho_e u^\alpha$ represents the electric current associated with the motion of the continuum relative to the observer's coordinate system.

In accornace with the definition (1.3) of the mass density ρ and the definition (1.4) of the free electrical charge density ρ_e of the medium, the scalars ρ and ρ_e satisfy, in the observer's coordinate system, the four-dimensional equations of continuity

$$\nabla_i(\rho u^i) = 0, \qquad \nabla_i(\rho_e u^i) = 0$$

In the Lagrangian (1.2) the four-dimensional invariant $-(16\pi)^{-1}F_{ij}F^{ij}$ is the Lagrangian of the electromagnetic field and the term $^1/_2\, F_{ij}P^{ij}$ determines the interaction between the electromagnetic field in the medium and the medium's polarization and magnetization intensity.

Let us determine the variations of the defining parameters of the model consistent with the definitions of variations adopted in [1, 2]. (The vector and tensor variations introduced below are transformed from one coordinate system to the other by the same laws which govern the transformation of respective vectors and tensors undergoing variations)

$$\begin{aligned}
\delta x^i &= x^{i\prime}(\xi^k) - x^i(\xi^k)\\
\delta A_k &= A_k'(\xi^k) - A_k(\xi^k)\\
\delta\pi^{ij} &= \pi^{ij\prime}(\xi^k) - \pi^{ij}(\xi^k)\\
\delta S &= S'(\xi^k) - S(\xi^k)
\end{aligned}$$

In this case variations of the remaining quantities appearing in the basic variational equation (1.1) are expressed in terms of parameter variations introduced above by the following relations:

$$\begin{aligned}
\delta F_{ij} &= \nabla_i \delta A_j - \nabla_j \delta A_i - \nabla_k A_j \nabla_i \delta x^k + \nabla_k A_i \nabla_j \delta x^k\\
\delta x_j^{\ i} &= x_j^{\ s} \nabla_s \delta x^i\\
\delta u^j &= g_k^{*j} u^i \nabla_i \delta x^k\\
\delta \rho &= -\rho g_k^{*i} \nabla_i \delta x^k\\
\delta P^{ij} &= \rho\delta\pi^{ij} - P^{ij} g_k^{*s} \nabla_s \delta x^k\\
\delta d\tau_4 &= d\tau_4 \nabla_i \delta x^i\\
g^{*ij} &= g^{ij} - u^i u^j
\end{aligned}$$

The raising and lowering of indices is performed everywhere with the use of the metric tensor of the observer's coordinate system with the covariant components g_{ij}, and $K_B(\xi^\mu)$ are the physical constants which define properties of the medium (such as anisotropy, dielectric permeability, etc.).

We further assume that $\delta K_B = 0$. The functional δW^* is chosen in the form

$$\delta W^* = \int\limits_{V_4} \{\rho T\delta S + j^k\delta A_k + j^k A_i\nabla_k\delta x^i - F_i\delta x^i\}\, d\tau_4$$

$$j^k = i^k + \rho_e^- u^k$$

where j^k are the contravariant components of the four-dimensional electric field vector and i^α are the components of the electrical conduction current.

The form of the functional δW^* was specified on the following considerations.

1°. The functional δW^* must include terms representing the work done by the volume forces F_α over possible displacements which are outside the system electromagnetic field-medium. This work is defined by the term $F_\alpha\delta x^\alpha$ which in δW^* appears with a minus sign. The reason for the minus sign there is that vector components in the four-dimensional pseudo-Euclidean Minkowski space, for which we shall use the notation $(- - - +)$ with the metric given above, are related to those in the three-dimensional space in the case when the coordinate system is Cartesian, by the following expressions (the notation $(+ + +)$ means that the vector components are computed relative to a coordinate system in a three-dimensional Euclidean space):

$$F^\alpha_{(- - - +)} = F_{\alpha(+++)} = - F_{\alpha(- - - +)}$$

and consequently

$$F_{\alpha(+++)}\,\delta x^\alpha = - F_{\alpha(- - - +)}\,\delta x^\alpha$$

Moreover we shall assume that the external (relative to the system electromagnetic field-medium) mass mements are absent (otherwise the expression for δW^* would have to contain a term representing the work done by the external mass moments on the possible rotations). The three- and four-dimensional components of the vector A_k have the same properties.

2°. The functional δW^* includes the term $F_4\delta x^4$, representing a possible external flux of energy other than heat, to the system electromagnetic field-medium. The expression for δW^* also includes electric current terms which depend on the magnitude of the uncompensated heat increment dQ' due to the dissipation effects.

Performing the variations in (1.1), taking into account (in accordance with (1.2)) the assumption made about the arguments of the internal energy and assuming that the variations δA_i, $\delta\pi^{ij}$, δS and δx^i are independent, we obtain Euler equations in the form of Maxwell equations for the electromagnetic field within the medium

$$\nabla_j H^{kj} = 4\pi j^k \qquad H^{ij} = F^{ij} - 4\pi P^{ij} \tag{1.5}$$

equations of state for the electromagnetic field in the medium and the temperature

$$F_{ij} = \frac{\partial U}{\partial \pi^{ij}}, \qquad T = \frac{\partial U}{\partial S} \tag{1.6}$$

and equations of momentum

$$\nabla_k \mathbf{P}_i{}^k = F_i \tag{1.7}$$

In the observer's coordinate system the following formulas representing the generalized equations of state are valid for the components of the total energy-momentum tensor of the system field-medium (1.8)

$$\mathbf{P}_i{}^k = S_i{}^k - \frac{1}{4} F_{pq}P^{pq}\delta_i{}^k + \frac{1}{2} F_{pq}P^{pq}g_i^{*k} + \rho\frac{\partial U}{\partial x_j{}^i} x_j{}^k - \rho^2\frac{\partial U}{\partial \rho} g_i^{*k} + \rho U u_i u^k$$

$$S_i^{\ k} = -\frac{1}{4\pi}\left(H^{kp}F_{ip} - \frac{1}{4}H^{pq}F_{pq}\delta_j^{\ k}\right)$$

Here the derivative $\partial / \partial x_j^{\ i}$ is taken with respect to the arguments $x_j^{\ i}$ appearing in the expression for the internal energy independently of the density ρ, and $S_i^{\ k}$ are components of the energy-momentum tensor of the electromagnetic field in the Minkowski space.

Computations made in the process of varying (1.1) show that in the region of continuous motions the following formula holds for the functional δW:

$$\delta W = \int_{\Sigma_3}\left\{\mathbf{P}_i^{\ k}\delta x^i + \frac{1}{4\pi}H^{ki}(\delta A_i + A_p\nabla_i\delta x^p)\right\}n_k\, d\sigma_3$$

where n_k is the four-dimensional unit vector normal to the three-dimensional surface Σ_3 bounding the volume V_4.

The system (1.5) – (1.8) is established on the assumption that the internal irreversible effects determined by the expression for functional δW^*, depend only on the presence of the electric current. When irreversible effects produced, e.g. by the irreversibility of the process of medium deformation or by the irreversibility of the magnetization process are considered, it is necessary to include in the expression for δW^* additional terms describing these effects. To close the derived system of equations we must introduce a relation representing the Ohm's law or its generalization which, e.g., for an isotropic medium, may have the form

$$i^k = -\sigma F^{ki}u_i + \alpha F^{kp}F_{pi}u^i$$

where σ is the coefficient of electrical conductivity of the medium and α is a coefficient determining the Hall effect.

2. The electromagnetic field energy-momentum tensor and the ponderomotive forces. We adopt, after Minkowski, the following definition for the components of the ponderomotive force vector

$$F_M^{\ \alpha} = -\nabla_j S^{\alpha j} \tag{2.1}$$

In this case the tensor components of the four-dimensional ponderomotive volume momentum excercised by the electromagnetic field on the medium are given by [1]

$$h^{ij} = -(S^{ij} - S^{ji}) \tag{2.2}$$

In the four-dimensional Cartesian coordinate system with the metric given above we have $S^{\alpha\beta} = S_{\alpha\beta}$, $S^{\alpha 4} = -S_{\alpha}^{\cdot 4}$ and $S^{4\alpha} = -S_{\cdot\alpha}^{4\cdot}$. We also note that all relations obtained in Sect. 2 are valid only in the intrinsic coordinate system, by which we understand the inertial coordinate system, and are chosen for each point M (ξ^1, ξ^2, ξ^3) of the moving continuum so that at each instant of time τ the three-dimensional velocity v of the point M is equal to zero in that coordinate system.

As we know [1, 3] the Minkowski electromagnetic field energy-momentum tensor and the Abraham energy-momentum tensor are linked, in the Cartesian inertial coordinate system relative to which the medium is at rest, by the following formulas:

$$A^{\alpha\beta} = \frac{1}{2}(S^{\alpha\beta} + S^{\beta\alpha}) \tag{2.3}$$

$$A^{\alpha 4} = A^{4\alpha} = S^{4\alpha}, \qquad A^{44} = S^{44}$$

from which it follows that the energy fluxes from the electromagnetic field to the medium computed in accordance with both the Minkowski and the Abraham hypotheses, are the same

$$F_A^{\ 4} = F_M^{\ 4} \tag{2.4}$$

For the components of the ponderomotive force tensor we assume, in accordance with the Abraham hypothesis, that

$$F_A^{\ \alpha} = -\nabla_j A^{\alpha j} \tag{2.5}$$

Then from (2.1) and (2.5) it follows that in the Cartesian coordinate system the following relations are valid:

$$F_A^{\ \alpha} = F_M^{\ \alpha} - \frac{1}{2}\nabla_j h^{\alpha j} - \frac{1}{2}\frac{\partial}{\partial t} h^{\alpha 4}$$

where the derivative with respect to time is taken in the intrinsic coordinate system. The formula defines the relation between components of vectors of the ponderomotive body force computed in accordance with the Abraham and Minkowski hypotheses.

As we already indicated, we shall further consider the models of media in the case when

$$x_j^{\ i},\ \pi^{ij},\ \rho,\ g_{ij},\ S,\ K_B$$

are used as the internal energy arguments. If we assume that the combined energy-momentum tensor of the medium and field is symmetric, i.e.

$$\mathbf{P}^{ik} = \mathbf{P}^{ki} \tag{2.6}$$

then the stated condition is equivalent to the four-dimensional momentum equation being identically satisfied under the condition that the sums of the characteristic intrinsic internal moments of the field and the medium are constant, or vanish along the world line. This does not, however, exclude the interaction between the field and the medium by means of the four-dimensional ponderomotive volume moments which may be produced by the asymmetry of the energy-momentum tensors of the field and the medium. If the fulfilment of condition (2.6) holds which, as shown below, restricts the form of dependence of the function of internal energy density on the arguments given above is not specified, then for the given arguments of the internal energy and for any arbitrary dependence of the internal energy function on its arguments, it is necessary to consider the momentum equations which can be used to determine the changes in the internal momenta of the medium.

Let us consider the following decompositions of the combined tensor of the energy-momentum of the field and medium into the energy-momentum tensor of the field and the energy-momentum tensor of the medium, according to Minkowski and Abraham, respectively

$$\mathbf{P}^{ik} = T_M^{\ ik} + S^{ik} = T_A^{\ ik} + A^{ik}$$

Here, in accordance with equality (1.8), T_M^{ik} and $T_A^{ik} = T_A^{ki}$ are the known Minkowski and Abraham energy-momentum tensors of the medium. Using (1.8) and (2.3) we obtain, respectively,

$$T_M^{\ ik} = -\frac{1}{4}F_{pq}P^{pq}g^{ik} + \frac{1}{2}F_{pq}P^{pq}g^{*ik} - \rho^2\frac{\partial U}{\partial \rho}g^{*ik} + \rho\frac{\partial U}{\partial x_j^{\ p}}x_j^{\ k}g^{ip} + \rho U u^i u^k \tag{2.7}$$

$$T_A^{\ ik} = \mathbf{P}^{ik} - A^{ik} = T_M^{\ ik} + \Omega^{ik} \quad (\Omega^{ik} = S^{ik} - A^{ik}) \tag{2.8}$$

In the general case for the same combined energy-momentum tensor of the electromagnetic field and medium with components P^{ik}, from condition of symmetry (2.6) and the relations (2.3) and (2.8) follows the inequality

$$T_M^{(ik)} + \Omega^{(ik)} = T_A{}^{ik} \neq T_M^{(ik)}$$

From (2.2) together with the particular fundamental assumption (2.6) and (2.7), it follows that for the specified internal energy arguments in the case when the electromagnetic field is assigned the Minkowski energy-momentum tensor, the specified function of the internal energy density must satisfy the relations

$$\rho \frac{\partial U}{\partial x_j{}^p} (x_j{}^k g^{ip} - x_j{}^i g^{kp}) = h^{ik}$$

which, by using the equations of state (first relation of (1.6)), can be rewritten in the form

$$\frac{\partial U}{\partial x_j{}^p} (x_j{}^k g^{ip} - x_j{}^i g^{kp}) = \frac{\partial U}{\partial \pi^{np}} (\pi^{ip} g^{nk} - \pi^{kp} g^{in}) \qquad (2.9)$$

If Eq (2.9) does not hold, the combined energy-momentum tensor P^{ik} of the electromagnetic field and medium is asymmetric, which in the general case, implies the presence in the system field-medium of four-dimensional, intrinsic momenta which vary along the world lines. We note that the tensor equation (2.9) can be written in the form of a system of six independent equations if the pair of free indices (i, k) passes through the values: (1,2), (1,3), (2,3), (1,4), (2,4) and (3,4) and if at the same time this system of partial differential equations is involutory.

3. Some implications following from the assumption of symmetry of the combined tensor of the electromagnetic field and medium energy-momentum. The tensor equation (2.9) obtained on the assumption that the combined tensor of the electromagnetic field and continuum energy-momentum (2.6) is symmetric, imposes restrictions on the form of the dependence of the function of internal energy density on its arguments, i.e. restrictions on the form and number of constants K_B (ξ^μ) specifying the geometrical and physical properties of the medium. Below we consider models of certain continua satisfying (2.9).

3.1. Let us consider the model of a medium defined by the form of the internal energy and assume that the internal energy constants K_B (ξ^μ) contain only one tensor G^0 with covariant components $g_{ij}{}^\circ = g_{ij}{}^\circ$ (ξ^k) which is the metric tensor of the initial state and may, in particular, coincide with the metric tensor of the observer's system with all remaining constants K_B being scalars (such a medium may be called isotropic).

With the above assumptions concerning the internal energy constants K_B we find that for the given set of the internal energy arguments, the components of the antisymmetric tensor π^{ij} yield only the following two, functionally independent invariants, which are solutions of (2.9):

$$\pi^{ij}\pi^{kn}g_{ik}g_{jn}$$

$$\pi^{ik}\pi^{gm}\pi^{rs}\pi^{np}g_{ir}g_{kn}g_{qs}g_{mp}$$

Introducing the four-dimensional symmetric "deformation" tensor E described in [2] with its covariant components defined by

$$E_{ij} = {}^1/_2 (x_i{}^m x_j{}^n g_{mn} - g_{ij}{}^\circ)$$

we can ascertain by direct test that any scalar function of the tensor components E_{ij}

is a solution of equation (2.9). It can also be directly verified that any scalar function of the second rank tensor $\mathbf{R}$ symmetric with respect to indices p and q

$$R_{qp} = R_{pq} = \pi^{ij}\pi^{ks}x_p{}^m x_q{}^n g_{im}g_{kn}g_{js}$$

is a solution of the tensor equation (2.9). Using the assumptions made about constants K_B (ξ^μ) we find that the components of tensors $\mathbf{E}$ and $\mathbf{R}$ can be formed into twelve independent invariants which can be used as the arguments of the internal energy. Thus the function of the internal energy density which satisfies (2.9) represents an arbitrary function of two invariants of the polarization-magnetization tensor, of the invariants of the tensors $\mathbf{E}$ and $\mathbf{R}$, of the density ρ and of the entropy S. If we restrict ourselves to the following set of arguments of the internal energy density: E_{ij}, π^{ij}, ρ, g_{ij}, S and K_B, then from equation (2.9) we directly obtain the equality $h^{ij} = 0$.

3.2. As a particular case of the general model of an elastic body with polarization and magnetization effects taken into account, we consider the model of a continuum whose internal energy density depends on the following arguments:

$$u^j,\ \pi^{ij},\ \rho,\ S,\ g_{ij},\ K_B$$

For this set of arguments the expression for the combined tensor of the electromagnetic field-medium energy-momentum is of the form

$$\mathbf{P}^{ik} = S^{ik} - \frac{1}{4} F_{pq}P^{pq}g^{ik} + \frac{1}{2} F_{pq}P^{pq}g^{*ik} + \rho \frac{\partial U}{\partial u^j} u^k g^{*ij} - \rho^2 \frac{\partial U}{\partial \rho} g^{*ik} + \rho U u^i u^k \tag{3.1}$$

In this case the assumption that the combined energy-momentum tensor (3.1) is symmetric, leads to the following differential tensor equation for the function of the internal energy density

$$\frac{\partial U}{\partial u^j}(u^k g^{ij} - u^i g^{jk}) = \frac{\partial U}{\partial \pi^{np}} (\pi^{ip}g^{nk} - \pi^{kp}g^{ni})$$

Assuming that the constants K_B do not include any tensors, we can show that the internal energy density is an arbitrary function of the following invariants formed from the tensor arguments π^{ij} and u^j:

$$u^i u^j g_{ij}, \qquad \pi^{ij}\pi^{kn}g_{ik}g_{jn}$$

$$\pi^{ik}\pi^{qm}\pi^{rs}\pi^{np}g_{ir}g_{kn}g_{qs}g_{mp}$$

$$\pi^{ij}\pi^{kp}u^m u^n g_{im}g_{kn}g_{jp}$$

as well as of the density ρ, and entropy S. We further assume that the function U of the internal energy density is a quadratic form in the polarization-magnetization tensor π (i.e. terms of the order of smallness higher than second can be neglected)

$$U = \rho \frac{4\pi\mu}{\mu - 1} \pi^{ij}\pi^{qk}g_{iq}g_{jk} + \rho \frac{2\pi(1 - \varepsilon\mu)}{(\varepsilon - 1)(\mu - 1)} \pi^{ij}\pi^{ql}u^m u^n g_{im}g_{qn}g_{jl} + f(\rho,\ u^i u^j g_{ij},\ S,\ K_B)$$

Here μ is the magnetic permeability coefficient and ε is the dielectric permeability and they can either be constants, or dependent on, e.g., temperature T and density of the medium ρ, and f is an arbitrary function of the arguments shown. The first two terms contain ρ as a multiplier. This is due to the choice of the usual notation for the internal energy arguments, which include the polarization-magnetization tensor π computed for a unit mass of the medium.

With the function of internal energy chosen in this manner, the equations of state for the electromagnetic field in the medium assume the form

$$F_{ij} = \frac{1}{2}\rho\left\{\frac{4\pi\mu}{\mu-1}(g_{ik}{}^*g_{jp}{}^* - g_{ip}{}^*g_{jk}{}^*) + \frac{4\pi}{1-\varepsilon}(g_{ik}u_ju_p - g_{jk}u_iu_p + g_{jp}u_iu_k - g_{ip}u_ju_k)\right\}\pi^{kp}$$

In the intrinsic coordinate system these relations reduce to

$$\mathbf{B} = \mu\mathbf{H}, \qquad \mathbf{D} = \varepsilon\mathbf{E}$$

Here $\mathbf{B}$ and $\mathbf{D}$ are the three-dimensional magnetic and electric induction vectors, respectively, while $\mathbf{H}$ and $\mathbf{E}$ are the magnetic and electric field intensity vectors computed in the intrinsic coordinate system.

3.3. Let us consider the models of the continua in which the set of constant parameters K_B includes the tensors describing the anisotropic properties of the material. As an example, we shall consider the continua possessing piezoelectric properties, determined by the mixed quadratic terms of the deformation tensor and the internal energy polarization-magnetization tensor, i.e. by terms of the form

$$D_{mn}^{ij}E_{ij}\pi^{mn} \tag{3.2}$$

Here we assume that the internal energy arguments are

$$E_{ij},\ \pi^{ij},\ \rho,\ g_{ij},\ S,\ K_B$$

First we consider the tensor equation (2.9) which holds in any inertial coordinate system. In particular (2.9) holds in the intrinsic coordinate system in which the internal energy arguments are determined with respect to an intrinsic coordinate system, which may, e.g., be Cartesian without loss of generality. When the inertial coordinate system is chosen in this manner we find that for the media with piezoelectric properties only, in the intrinsic coordinate system we have $\pi^{\alpha\beta} = 0$.

Let us establish how many linearly independent tensor coefficients

$$K_{ij} = -K_{ji} = D_{ij}{}^{mn}E_{mn}$$

are admitted by (2.9). Substituting the function (3.2) into (2.9), we obtain a system of six linear homogeneous equations for the six components of the antisymmetric tensor K_{ij}. The rank of the matrix of the principal determinant of such a system is four, therefore the components of the tensor K_{ij} contain just two linearly independent components.

In a similar manner we can obtain the number of linearly independent components in the tensor coefficients accompanying the higher order terms with respect to the polarization-magnetization tensor.

This work was supervised by L. I. Sedov to whom the author is deeply grateful.

REFERENCES

1. Sedov, L. I., A Course in Continuum Mechanics. Vol. 1, Wolters-Noordhoff, 1971.
2. Sedov, L. I., Mathematical methods of constructing new models of continua. Uspekhi matem. nauk, Vol. 20, №5, 1965.

3. Pauli, W., Theory of Relativity. Pergamon Press, Book №13129, 1958.

Translated by L.K.

UDC 538.4

SIMPLE WAVES IN NONCONDUCTING MAGNETIZABLE MEDIUM

PMM Vol.37, №5, 1973, pp.813-821
I.E.TARAPOV
(Khar'kov)
(Received February 26,1973)

The propagation of small amplitude waves through a nonconducting, isotropically magnetizable medium is studied, and simple wave equations obtained. Simple waves in an ideal magnetizable gas are studied in detail. The problem of stability is considered for the ideal gas and a magnetizable fluid, and the parameter values for which the wave phase velocities become imaginary are determined.

The motion of a medium which does not conduct current but can be isotropically and nonuniformly magnetized in an external magnetic field, can be described by the following system of equations [1]

$$\frac{\partial\rho}{\partial t}+\operatorname{div}\rho\mathbf{v}=0,\qquad \rho T\frac{d}{dt}(s+s^*)=\tau_{ik}\frac{\partial v_i}{\partial x_k}+\lambda^\circ\Delta T \tag{1}$$

$$\rho\frac{d\mathbf{v}}{dt}+\nabla(p+\psi)-M\nabla H=\eta_1\Delta\mathbf{v}+\left(\eta_2+\frac{1}{3}\eta_1\right)\nabla\operatorname{div}\mathbf{v}$$

$$\operatorname{div}\mathbf{B}=0,\qquad \operatorname{div}\mathbf{E}=0$$

$$\frac{\partial\mathbf{B}}{\partial t}=-c\operatorname{rot}\mathbf{E},\qquad \varepsilon\frac{\partial\mathbf{E}}{\partial t}=c\operatorname{rot}\mathbf{H}$$

$$\mathbf{B}=\mathbf{H}+4\pi M(\rho,T,H)\mathbf{H}/H,\qquad p=p(\rho,s),\qquad T=T(\rho,s)$$

$$\left(\psi=\int_0^H\left\{M-\rho\left(\frac{\partial M}{\partial\rho}\right)_{T,H}\right\}dH,\qquad s^*=\frac{1}{\rho}\int_0^H\left(\frac{\partial M}{\partial T}\right)_{\rho,H}dH\right)$$

Here τ_{ik} is the viscous stress tensor; λ°, η_1 and η_2 are constant coefficients of heat conductivity, first and second viscosity, respectively; $M(\rho,T,H)\equiv(4\pi)^{-1}(\mu-1)H$ is a function of magnetization (assumed known), $\mu=\mu(\rho,T,H)$ is the magnetic permeability of the medium, the dielectric permeability ε is constant and free charges are absent.

The propagation of small amplitude waves in such a medium can be described by the following system of seven equations:

$$\frac{\partial u_i}{\partial t}+x_{ik}\frac{\partial u_k}{\partial x}=d_{ik}\frac{\partial^2u_k}{\partial x^2}\qquad(i,k=1,2,\ldots 7) \tag{2}$$

$$u_1\equiv\rho',\ u_2\equiv s',\ u_3\equiv v_x',\ u_4\equiv B_y',\ u_5\equiv B_z',\ u_6\equiv E'_y,\ u_7\equiv E_z'$$

Here u_i denote perturbations of the variables and the matrices $\|x_{ik}\|$ and $\|d_{ik}\|$ have the following nonzero components: